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Planck distribution in *n*-dimensions : Conceptual and practical applications

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Abstract The pioneering work of Landsberg and De Vos on black-body radiation in n spatial dimensions is extended on four fronts (i) a new n-dependent recipe is given to count the spin-degeneracy factor of the photon, and the vectorial/tensorial structure of the underlying Maxwell fields is exhibited, (ii) the role of the Stefan-Boltzmann constant σ_1 is re-examined for linear resistors, (iii) the efficiency/efficacy of luminous objects radiating in n-dimensions is calculated, (iv) finally, our results are linked to the law of degradation of energy in context of the early universe

Keywords Planck radiation, n dimensions, efficiency and efficacy.

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1. Introduction

It is now-a-days well recognised that the concept of "dimensions" plays an important role in the theory of distribution functions, phase transitions, fractal growth, field theoretic renormalisation, superstring quantisation, *etc*. In this context, a beautiful paper by Landsberg and De Vos [1] (abbreviated as LD hereafter) on the Stefan-Boltzmann constant σ_n in n-dimensional space becomes particularly relevant. By a judicious combination of hyperspace geometry and Bose-Einstein statistics these authors derived

$$\sigma_n = r_n \frac{\pi^{(n-1)/2}\Gamma(n+1)\zeta(n+1)k^{n+1}}{\Gamma[(n+1)/2]h^2c^{n-1}}, \qquad (1)$$

where r_n is the spin-degeneracy factor of the photon, Γ the gamma function, ζ the Riemann zeta function, k the Boltzmann constant, h the Planck constant, and c the speed of light. The aim of the present paper is to extend further the work of LD by focussing attention on the following four issues :

(i) LD have taken the number of independent polarization states of the photon as two irrespective of the space dimension. We examine in Section 2 below the possibility of r_n being dimension-dependent together with the tensorial structure of the underlying electromagnetic fields.

(ii) LD have already discussed several implications of σ_n with regard to practical devices operating in low dimensions as well as quantum field theories constructed in high dimensions. In Section 3 we shall consider an interesting implication of σ_1 relevant to photons in electrical networks.

(iii) LD have used σ_n to evaluate the intensity of black-body radiation from a hypersphere of radius R. In Section 4 we shall use the generalized Planck distribution in the visible region to evaluate the efficiency [*cf.* eq. (6)] of luminous objects in n-dimensions. In Section 5 the concept of the spectral response of the eye is also brought in to evaluate the efficacy [*cf.* eq. (16)] of a filament radiating in n-dimensions.

(iv) Finally in Section 6, some concluding remarks are given on the possible link between our results and the law of degradation of energy in the early universe.

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2. Spin-degeneracy factor of a photon

It is well known that in the usual 3-dimensional space, a monochromatic plane electromagnetic wave can be described by a vector potential $\boldsymbol{A}$, electric field $\boldsymbol{E} = -\partial \boldsymbol{A} / \partial t$ (with t being the time), and magnetic field $\boldsymbol{H} = \nabla \times \boldsymbol{A}$. There are two possible transverse states of polarization as measured by the direction of the electric vector. LD assumed this to be the case also for photons moving in arbitrary spatial dimension n, and chose

$$r^{\mathrm{LD}} = 2 \tag{2}$$

However, it is possible to count r_n via a different philosophy which is crucially n-dependent. Indeed in $n = 1$ dimension the concept of spin does not exist implying that the corresponding wave equation is the massless scalar one. Based on this argument Menon and Agrawal [2] (abbreviated as MA) chose

$$r_1^{\mathrm{MA}} = 1, \; n = 1 \tag{3a}$$

in $n \geq 2$ dimensions, however, the concept of spin does exist implying that the corresponding wave equation is the massless vector one *viz*.

$$\left(\nabla^{(n)2} - \partial^2 / c^2 \partial t^2\right) A_i = 0; \; i = 1, 2 \ldots n \tag{3b}$$

subject to the subsidiary condition $\sum_{i=1}^{n} \partial A_i / \partial x_i = 0$. Here, $\nabla^{(n)2}$ is the n-dimensional Laplacian operator and A_i stands for the i-th Cartesian component of the space-time-dependent vector potential. The electric field $\boldsymbol{E}$ now becomes an n-component vector but the magnetic field becomes an $n(n-1)/2$ component tensor given by

$$E_j = -\partial A_j / \partial t; \; 1 \leq j \geq n,$$

$$H_{ij} = \partial A_j / \partial x_i - \partial A_i / \partial x_j; \; i < j. \tag{3c}$$

Clearly, a monochromatic plane electromagnetic wave travelling along a given axis can have its polarization vector (*i.e.*, the direction of the electric field) along any of the remaining $n - 1$ Cartesian axes. Hence, according to Menon and Agrawal [2], the relevant photon will have

$$r_n^{\mathrm{MA}} = n - 1 \; : n \geq 2. \tag{3d}$$

Although the value of the Stefan-Boltzmann constant together with the absolute radiant flux emitted by a surface will be altered by the prescription [eqs. (3a,d)] yet the curves of the normalised Planck spectrum drawn in Ref. [1] will remain unaffected. Before leaving this section, however, an important remark must be made on the question of the polarization degrees of freedom of a photon moving in n spatial $+1$ temporal dimensions. A satisfactory resolution of this question depends on an analysis of the little group of the lightlike vector $(0, 0, \ldots, 0, 1, 1)$ in $n + 1$ dimensions. The machinery for doing this in $3 + 1$ dimensional Minkowski space is standard, following the work of Wigner on the representations of the Lorentz group [3]. However in the general case, this problem appears quite complicated; so we directly count the number of independent degrees of freedom of the gauge potential.

3. Previous implications of σ_n

LD have already given a beautiful exposition of the possible relevance of σ_1 in the context of practical devices operating in low dimensions (*e.g.* σ_1 in electrical networks, σ_2 in planar photonic devices, *etc*). They have also hinted that σ_n may play a role in the supersymmetric field theories formulated in $n = 9$ or 25 flat spatial dimensions.

Here, we wish to make an interesting observation on one of these previous implications *viz.* the thermal noise in a linear resistor. Using the Menon-Agrawal prescription (3) for σ_1 along with a black-body emissivity $\epsilon_{\text{black}} = 1$ one finds for the radiant power ϕ_1 emitted per unit area

$$\phi_1^{\mathrm{MA}} = \pi^2 k^2 T^2 / 6h. \tag{4}$$

It is worth recalling that in order to arrive at this result employing their recipe (2), LD had to assign a grey-body emissivity $\epsilon_{\text{grey}} = 1/2$ to linear resistors.

4. Efficiency of luminous objects

4.1. Preliminaries :

LD had made an interesting application of their theory to calculate the full radiant flux Φ emitted by a hypersphere of radius R embedded in n-dimensions and found (their Figure 4) that for $RT >> hc/2\pi k$, Φ was an increasing function of n. Here, we wish to supplement their work by focussing attention on the estimated efficiency (defined below) of hot luminous bodies.

4.2. Analogies from 3-dimensional case :

In order to fix ideas let us recapitulate some properties of the light emission from the filaments of incandescent lamps operating in $n = 3$ dimensions. Let λ be the wavelength and $\nu = c/\lambda$, the frequency of the wave. The visible region is usually taken as $\lambda_v \leq \lambda \leq \lambda_r$ where the violet and red wavelengths are

$$\lambda_v = 380 \text{ nm}, \; \lambda_r = 760 \text{ nm}. \tag{5}$$

4.3. Previous results on 3-dimensional efficiency :

In Ref. [4], the efficiency of a filament at temperature T was defined as

$$\eta_n(T) = \frac{\text{Power emitted in the visible region}}{\text{Total input power}}. \tag{6}$$

A detailed numerical analysis of the efficiency of commercial incandescent lamps operating in 3-dimensions was carried out by Agrawal *et al* [4] over a wide temperature range. They found that $\eta_3(T)$ was almost independent of the actual bulb design, filament geometry, coupling to loss channels, and the detailed wavelength-dependence of the grey body emissivity function $\epsilon_{\text{grey}}(\lambda, T)$. In otherwords, $\eta_3(T)$ was found to be an essentially universal function of T, calculable using the black-body emissivity $\epsilon_{\text{black}} =$, vanishing both for

$T \to 0$ and $T \to \infty$ and developing a theoretical peak value $\eta_3(T) = 0.274$ at $T = 6989$ K (which incidentally is much above the melting point of tungsten metal of which the filaments are composed).

4.4. Formulation of $\eta_n(T)$ in n-dimensions :

Referring to the work of LD [their eqs.(7–9)], we write the Planck law in the form

$$I(\lambda,T)d\lambda = r_n \frac{\pi^{(n-1)/2} hc^2 A}{\Gamma[(n+1)/2]} \cdot \frac{d\lambda}{[\exp(hc/kT\lambda)-1]}, \quad (7)$$

where $I(\lambda, T)\, d\lambda$ is the electromagnetic power radiated between the wavelengths λ and $\lambda + d\lambda$, and A is the surface area of the emitting body. Then the efficiency still defined by eq. (6) becomes

$$\eta_n(T) = \int_{\lambda_v}^{\lambda_r} I(\lambda,T)d\lambda \Big/ \int_0^{\infty} I(\lambda,T)d\lambda$$

$$= \frac{1}{\Gamma(n+1)\zeta(n+1)} \int_{\lambda_v}^{\lambda_r} \frac{d\lambda}{\lambda^{n+2}[\exp(hc/kT\lambda)-1]}, \quad (8)$$

where the limits λ_v and λ_r are assumed to be the same as in 3-dimensional case.

4.5. Transformation of variable :

Eq. (8) may be cast into an elegant, compact form by introducing

$$x = kT\lambda / hc, \; d\lambda = hc\,dx / kT. \quad (9)$$

Inserting back into eq. (8) we find

$$\eta_n(T) = \frac{1}{\Gamma(n+1)\zeta(n+1)} \int_{x_v}^{x_r} \frac{dx}{x^{n+2}[\exp(1/x)-1]}, \quad (10)$$

where $x_v = kT\,\lambda_v/hc$ and $x_r = kT\,\lambda_r/hc$. This integral is amenable to both theoretical and numerical analyses.

4.6 Algebraic properties of $\eta_n(T)$:

As shown in Appendix A, some interesting algebraic properties of $\eta_n(T)$ can be derived analytically if the dimensionsality is large enough. Letting

$$n >> 1;\; p_r = \frac{\ln(\lambda_r/\lambda_v)}{\lambda_r/\lambda_v - 1}, \quad (11)$$

we find [*cf.* eqs. (A7) and (A13)] that the location T^* and the value η^* of the peak in the efficiency are given by

$$T^* \approx \frac{hc}{kp_r\lambda_r}\frac{1}{n+1} \approx \frac{27300}{n+1} \quad (12)$$

$$\eta^* \approx 1 \text{ if } n \ge 8/(1-p_r)^2 \ge 85. \quad (13)$$

4.7 Numerical work :

We took a dimensionality range $1 \le n \le 6$ in steps of unity, and a temperature range $100 \le T \le 10000$ K in steps of 100 K. The cut-offs λ_v and λ_r were chosen to be same as in the 3-dimensional case. For n = 1, 2 and 3, this is justified since there are human beings to see the radiation; for $n \ge 4$ the only justification is simplicity. For each set, the right-hand-side of eq. (10) was evaluated by Simpson's quadrature. The results are shown in Figure 1. It is seen that, for fixed n, our

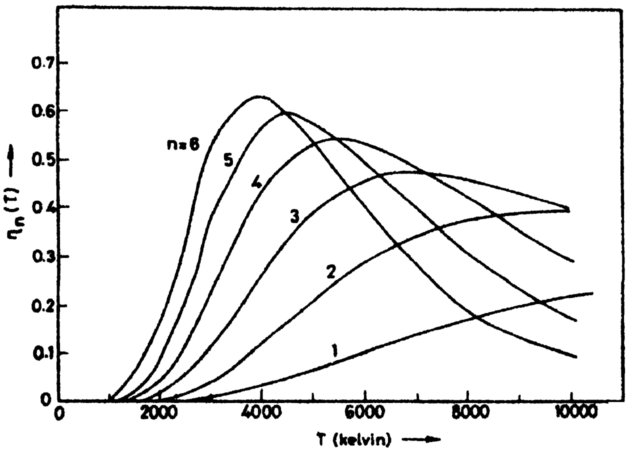

Figure 1. Variation of the efficiency $\eta_n(T)$ with temperature T [*cf.* eq. (10)] for the specified values of dimensionality n

$\eta_n(T)$ tends to zero both as $T \to 0$ and $T \to \infty$ so that a peak appears at some intermediate $T = T^*$. Table 1 lists the

Table 1. Numerical results for the peak values of the efficiency and the corresponding temperature at specified dimensions. The analytically estimated temperatures [cf. eq. (12)] are also displayed.

n	Numerical		Analytical
	η_n^*	T^*	T^*
1	0.267	17240	13650
2	0.391	9740	9100
3	0.475	7000	6825
4	0.538	5520	5460
5	0.588	4570	4550
6	0.629	3910	3900

numberical peak values and the corresponding temperature T^* for n = 1 to 6 obtained through a fine search. Clearly, the peak value $\eta_n(T^*)$ is a monotonically increasing function of n. For comparison the theoretically predicated T^* [*cf.* eq. (12)] is also mentioned in Table 1 and the agreement is found to be reasonable for $n \ge 3$. Recalling our experience with incandescent lamps, it is expected that the results of Figure 1/Table 1 should be essentially independent of the shape and greyness of the radiating body.

5. Efficacy of visible objects

5.1. Preliminaries :

Next we turn our attention to efficacy of visible objects borrowing ideas from 3-dimensional case to begin with. It is known that the spectral response function $S(\lambda)$ of the human eye becomes a maximum at the yellow wavelength

$$\lambda_m = 555 \text{ nm} \quad (14)$$

at which a radiated power of 1 watt provides a luminous flux of 683 lumens. At other wavelengths the number of lumens per watt is given by the quantity 686 $S(\lambda)$ which becomes vanishingly small outside the visible region $\lambda_v \leq \lambda \leq \lambda_r$. In Ref. [4], this $S(\lambda)$ was parameterised *via* least-squares fitting in the form

$$S(\lambda) = \exp(-az^2 + bz^3) \quad ,$$

$$\text{with} \quad z = \lambda / \lambda_m - 1 = x / x_m - 1, \tag{15}$$

$$a = 87.868,\ b = 40.951.$$

5.2. Previous results on 3-dimensional efficacies :

In Ref. [4], the efficacy of a filament at temperature T was defined as

$$e(T) = \frac{\text{Total luminous flux emitted}}{\text{Total radiant power emitted}} \cdot \tag{16}$$

The physical difference between the efficiency η and efficacy e becomes clear by a comparison of eqs. (6) and (16). The crucial role is played by the spectral response function $S(\lambda)$ of the eye. In the extreme case of $S(\lambda)$ being λ-independent both η and e would be identical. In the other extreme case of $S(\lambda)$ being a narrow Gaussian function, e would have almost vanished.

A detailed numerical analysis of the efficacy of commercial incandescent lamps operating in 3-dimensions was carried out by Agrawal *et al* [4] over a wide temperature range. They found that $e(T)$ was almost independent of the actual bulb design, filament geometry, coupling to loss channels, and the detailed wavelength-dependence of the grey body emissivity function $\epsilon_{\rm grey}(\lambda, T)$. In other words, $e(T)$ was found to be an essentially universal function of T, calculable using the black-body emissivity $\epsilon_{\rm black} =$, vanishing both for $T \to 0$ and $T \to \infty$, and developing a theoretical peak at 6625 K (which incidentally is much above the melting point of tungsten metal of which the filaments are composed).

5.3 Formulation of e_n (T) in n-dimensions :

Multiplying the integrand of $\eta_n(T)$ by 683 S is enough to turn eq. (10) into $e_n(T)$ and hence we get

$$e_n(T) = \frac{1}{\Gamma(n+1)\zeta(n+1)} \int_{x_v}^{x_r} \frac{683\, S\, dx}{x^{n+2}[\exp(1/x) - 1]} \cdot \tag{17}$$

Once again, it can be treated both analytically and computationally.

5.4. Algebraic properties of e_n (T) :

As demonstrated in Appendix B, some interesting mathematical properties of $e_n(T)$ can be derived analytically if the dimension n is large enough. From eqs. (B6) and (B8), we find that the maximum in the efficacy has location T' and value e_n' given by

$$T' \quad \frac{hc}{k\lambda_m} \frac{1}{(n+1)} \sim \frac{26000}{(n+1)}\ {}^\circ\text{K}, \tag{18a}$$

$$e_n' \approx 683\ (n / 2a)^{1/2}\ \text{L/W}, \tag{18b}$$

where the mean wavelength λ_m and shape parameter a describe the spectral response function $S(\lambda)$ [*cf.* eq. (15)] of the eye. Thus, e_n' becomes 683 LW^{-1} at a special value $n = 2a = 175$.

5.5. Numerical work :

Following the procedure of the previous section (*cf.* 4.7), the integral (17) was evaluated numerically and the results are shown in Figure 2 and Table 2. Here also the agreement between the exact computer results and simple theoretical estimates is good for $n \geq 3$.

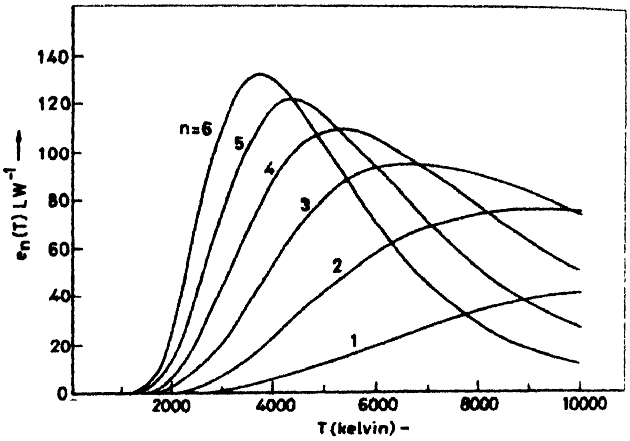

Figure 2. Variation of the efficacy $e_n(T)$ with temperature T [*cf* eq (17)] for the specified values of dimensionality n

Table 2. Numerical results for the peak values of the efficacy and the corresponding temperature at specified dimensions. The analytically estimated efficacy and temperatures [cf eqs (18a), (18b)] are also displayed.

n	Numerical		Analytical	
	e_n' LW^{-1}	T'	e_n' LW^{-1}	T'
1	50.9	16410	51.5	13000
2	76.1	9270	72.8	8670
3	94.4	6670	89.2	6500
4	109.0	5260	103.0	5200
5	121.1	4370	115.1	4330
6	131.6	3740	126.1	3710

6. Concluding remarks

The results of Sections 4 and 5 may have some interesting albeit speculative implications if extrapolated appropriately to the case of the early big-bang universe (ignoring quantum gravity complications).

According to superstring/supersymmetric field theory, around the Planck epoch, the spatial dimension n was quite large, perhaps more than 10 or 25, and the temperature also was very high *viz.* $T \approx 10^{32}$ K. The corresponding black-body spectrum would be peaked at the wavelength $\lambda_m \sim 10^{-33}$ cm which is the Planck length. As the epoch increased, n went on diminishing in analogy with the models of dimensional compactification [5]. (This analogy is not to be stretched too far because, while our speculation suggests a gradual reduction in n, the current models of dimensional compactification achieve it in one go). Then, as suggested by Figures 1 and 2, the efficiency and efficacy all went down.

From a physical viewpoint, the efficiency or efficacy is a measure of the useful (*i.e.* visible) fraction of the net thermal radiation emitted by the body. Of course, since there were no human beings present in the very early universe to *see* the thermal radiation, a more logical measure of the useful fraction would be

$$\bar{\eta} = \int_{\bar{\lambda}}^{\infty} I(\lambda, T) d\lambda \Big/ \int_0^{\infty} I(\lambda, T) d\lambda , \qquad (19)$$

where $\bar{\lambda}$ is an appropriate cut-off. However, in the discussion at hand we have used η and e since their analytical estimates/numerical graphs have already been obtained.

The cosmological expansion of the universe is accompanied by a dimensionality reduction together with a general degradation of energy. According to the second law of thermodynamics energy tends to pass from a more available to a less available form which is consistent with the above-mentioned decline of η or e as η gets reduced. This is also in conformity with the decline of the total black-body radiation from a hypersphere with the decrease of n as observed in Figure 4 of LD [1]

Acknowledgment

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Appendix A : Approximate analytical results on the efficiency function [*cf.* eq. (8)]

A.1. Transformation of variable :

It will be more convenient to use a dimensionless combination containing the frequency c/λ of radiation as the main variable and rewrite eq. (8) as

$$\eta_n(T) = G / H : H = \Gamma(n+1)\zeta(n+1),$$

$$G = \int_{y_r}^{y_v} \frac{y^n \, dy}{\exp(y) - 1} : y = \frac{1}{x} = \frac{hc}{kT\lambda}. \qquad (A1)$$

Below, we shall be interested only in those situations in which $y_v >> 1$ and $y_r >> 1$ so that we can approximate

$$G = \int_{y_r}^{y_v} y^n e^{-y} dy;\ y_v = \frac{hc}{kT\lambda_v},\ y_r = \frac{hc}{kT\lambda_r} \qquad (A2)$$

A.2. Peak with respect to temperature :

Since the temperature dependence of G appears only in the integration limits, hence

$$\frac{\partial G}{\partial T} = -\frac{1}{T}\{y_v^{n+1}\exp(-y_v) - y_r^{n+1}\exp(-y_r)\}. \qquad (A3)$$

For a fixed n, the maximum of G (labeled by a star*) appears when

$$\left.\frac{\partial G}{\partial T}\right|_{T^*} = 0;\ \exp(y_v^* - y_r^*) = \left(\frac{y_v^*}{y_r^*}\right)^{n+1}. \qquad (A4)$$

This can be cast into an elegant form by introducing the symbols

$$q = \frac{\lambda_r}{\lambda_v} > 1;\ p_r = \frac{\ln q}{q-1};\ p_v = \frac{q \ln q}{q-1} \qquad (A5)$$

Then eq. (A4) and its logarithm yield

$$\exp[y_r^*(q-1)] = q^{n+1},$$

$$y_r^* = (n+1)p_r : y_v^* = (n+1)p_v. \qquad (A6)$$

Recalling the definition (A2) of y_r^* *viz.* $hc/kT^*\lambda_r$ and equating it to the quantity $(n+1)\,p_r$ we obtain

$$T^* = \frac{hc}{k\lambda_r p_r}\frac{1}{(n+1)} = \frac{27300}{(n+1)}\ {}^\circ\mathrm{K} \qquad (A7)$$

which is the result quoted in the text, provided we use the values q = 760 nm/360 nm = 2; p_r = 0.693, p_v 1.386 as known in the 3-dimensional case.

A 3. Efficiency value at the peak :

Now at temperature T^*, we wish to evaluate the integral [*cf.* eq. (A2)]

$$G^* = \int_{y_r^*}^{y_v^*} \exp(-F) dy;\ F = y - n \ln y. \qquad (A8)$$

In view of the derivatives,

$$\frac{\partial F}{\partial y} = 1 - \frac{n}{y};\ \frac{\partial^2 F}{\partial y^2} = \frac{r}{y} \qquad (A9)$$

it is clear that F has a maximum at the point n. Employing a second-order Taylor expansion of F around n and intserting it back into eq. (A8) leads to

$$G^* = \int_{y_r^*}^{y_v^*} \exp\left[-(n - n\ln n) - \frac{(y-2)^2}{2n}\right] dy$$

$$= (2n)^{1/2} n^n \exp(-n) \int_{t_r^*}^{t_v^*} dt \exp(-t^2);\ t = \frac{y-n}{(2n)^{1/2}}. \qquad (A10)$$

Of course, such a Gaussian approximation for the integrand will be better the larger n is compared to unity.

A.4. Behaviour of efficiency at the peak :

In eq. (A1), we had defined $\eta_n(T) = G/H$ where $H = n!\ \zeta(n+1) \approx (2\pi n)^{1/2} n^n \exp(-n)$ by Stirling formula for $n >> 1$. This yeilds, in view of eq. (A10), the value of the efficiency at temperature T^* in terms of error function as

$$\eta_n^* = G_n^*/H \approx \frac{1}{2}\left[\mathrm{erf}(t_v^*) - \mathrm{erf}(t_r^*)\right] \qquad (A11)$$

with $t_v^{\bullet} = \dfrac{y_v^{\bullet} - n}{(2n)^{1/2}} \approx (n/2)^{1/2}(p_v - 1) > 0$,

$$t_r^{\bullet} = \frac{y_r^{\bullet} - n}{(2n)^{1/2}} \approx (n/2)^{1/2}(p_r - 1) < 0 . \quad \text{(A12)}$$

Here, $y_v^{\bullet}$ and $y_r^{\bullet}$ have been read-off from eq. (A6). Now erf $(t_v^{\bullet}) \to 1$ if $t_v^{\bullet} \approx 0.4\ (n/2)^{1/2} \geq 2$ and erf $(t_r^{\bullet}) \to -1$ if $t_r^{\bullet} = -0.3\ (n/2)^{1/2} \leq -2$.

Hence, $$\eta_n^{\bullet} \to 1 \text{ if } n \geq \frac{8}{(1-p_r)^2} \geq 85 \quad \text{(A13)}$$

as was claimed in the text [*cf.* eq. (13)], provided the values of p_v and p_r in 3-dimensional case are extrapolated into higher dimensions.

Appendix B : Approximate analytical results on the efficacy function [*cf.* eq. (17)]

B.1 Starting expression :

Let us rewrite eq. (17) in the form

$$e_n(T) = 683\ J/H,$$

$$J = \int_{\lambda_v}^{\lambda_r} \frac{\exp(-az^2)dx}{x^{n+2}[\exp(1/x) - 1]};\ x = \frac{kT\lambda}{hc}, \quad \text{(B1)}$$

$$a \approx 88;\ x_m = kT\lambda_m/hc;\ z = x/x_m - 1. \quad \text{(B2)}$$

Here, the spectral response function S has been approximated by a simple Gaussian *viz.* $\exp(-az^2)$, ignoring the small skewness introduced by the exp $[bz^3]$ term in eq. (15).

B.2. Further reduction :

Due to the sharp peak of the integrand at x_m, we can take out the remaining slowly-varying factors (evaluated at x_m) outside the integral and obtain

$$J = \frac{1}{x_m^{n+1}[\exp(1/x_m) - 1]} \int_{z_v}^{z_r} e^{-az^2} dz\ , \quad \text{(B3)}$$

where $z_r = \lambda_r/\lambda_m - 1 \approx 0.4$ and $z_v = \lambda_v/\lambda_m - 1 \approx -0.3$ correspond respectively, to the red and violet ends of the visible spectrum. Now, $a^{1/2} z_r \approx 3.7 > 3$, $a^{1/2} z_v \approx -2.8 < -2$ and the number $x_m << 1$ in the present discussion.

Hence with reasonable accuracy, eq. (B3) can be estimated by using $\int_{-\infty}^{\infty} \exp(-az^2)dz = (\pi/a)^{1/2}$ so that

$$J = y_m^{n+1} \exp(-y_m)(\pi/a)^{1/2};\ y_m = \frac{1}{x_m} = \frac{hc}{kT\lambda_m} . \quad \text{(B4)}$$

B.3. Peak with respect to temperature :

For fixed n, our J depends on T only through the variable y_m. Therefore, its maximum (labeled by a prime) appears at

$$\left.\frac{\partial J}{\partial T}\right|_{T'} = 0;\ y_m' = \frac{hc}{kT\lambda_m} = n + 1\ . \quad \text{(B5)}$$

This is readily solved for T' to give

$$T' = \frac{hc}{k\lambda_m(n+1)} = \frac{26000}{n+1}\ {}^{\circ}\mathrm{K}, \quad \text{(B6)}$$

as claimed in the text [*cf.* eq. (18a)] .

B.4. Formula for the peak efficacy :

At temperature T', our y_m' has reduced to $n + 1$.

Therefore, eq. (B4) gives

$$J = (n+1)^{n+1} \exp(-n-1)\ (\pi/a)^{1/2}$$
$$\approx n^{n+1} \exp(-n)\ (2\pi/2a)^{1/2} \text{ for } n >> 1. \quad \text{(B7)}$$

In the text [*cf.* eq. (B1)], the efficacy was defined as $e_n(T) = 683\ J/H$ where $H \approx (2\pi n)^{1/2}\ n^n \exp(-n)$ using Stirling's approximation for large n. Using eq. (B7) we get for the efficacy at the peak

$$e' = \frac{683J}{H} \approx 683\left(\frac{n}{2a}\right)^{1/2} \approx 56\ n^{1/2}\ \mathrm{LW}^{-1} \quad \text{(B8)}$$

which coincides with eq. (18b) of the text.